Substructural modal logic for optimality and games

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Resource Reasoning
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Overview

- Focus: logical characterisations of notions of optimality.
- Normal form games.
- Extensive form games.

Normal form games

Example (Prisoner's dilemma, normal form)

u	c	d
c	(-1,-1)	(-6,0)
d	(0,-6)	(-3,-3)

- *ab* means person does action *a*, and person 2 does action *b*
- (x, y) means person 1 gets x years in prison, and person 2 gets y years in prison.

Normal form games

Definition (Best response)

A choice (or action, or strategy) a is an agent's best response to another agent's choice b, if there is no choice c such that the (first) agent can perform such that the (first) agent prefers cb to ab.

Normal form games

- Action a is the best response to action b for payoff function v at world w if

$$w \models \forall \alpha. \begin{pmatrix} (\langle a \rangle \top \wedge \langle \alpha \rangle \top) * (\langle b \rangle \top) \\ \rightarrow \\ v(\alpha b) \leq v(ab) \end{pmatrix}.$$

  holds.
- We abbreviate this formula as $BR(a, b, v)$.

Normal form games

- In the prisoner's dilemma example PD, the payoff function v_1 for the first agent is:

$$\begin{array}{rclrcl} v_1(cc) & = & -1 & v_1(cd) & = & -6 \\ v_1(dc) & = & 0 & v_1(dd) & = & -3. \end{array}$$

- The first agent's best response to the second agent collaborating is to defect, and hence:

$$PD \models BR(d, c, v_1).$$

Normal form games

Definition (Concurrent transition system)

A concurrent transition system is a structure $(\mathbf{S}, \mathbf{Act}, \to, \circ, e)$ such that

- $(\mathbf{S}, \mathbf{Act}, \to)$ is a labelled transition system,
- $\circ : \mathbf{S} \times \mathbf{S} \rightharpoonup \mathbf{S}$ is concurrent composition operator, and
- $e \in \mathbf{S}$ is a distinguished element of the state space,

with various well-formedness conditions on the interaction of $\to$, e, and $\circ$.

Normal form games

- The semantics of modal operators and multiplicative conjunction are based on $\rightarrow$ and $\circ$:

$$
\begin{array}{lcll}
w & \models & \langle a \rangle \phi & \text{iff} \;\; \text{there exists } w \xrightarrow{a} w' \\
& & & \qquad \text{such that } w' \models \phi \\[1em]
w & \models & \phi_1 * \phi_2 & \text{iff} \;\; \text{there exist } w_1 \text{ and } w_2, \text{ where} \\
& & & \qquad w \sim w_1 \circ w_2, \text{ such that} \\
& & & \qquad w_1 \models \phi_1 \text{ and } w_2 \models \phi_2
\end{array}
$$

Normal form games

- Payoffs are functions from actions to $\mathbb{Q} \cup \{-\infty\}$.
- The term language includes arithmetic and payoff functions applied to actions.
- We quantify over both actions and numerical values.

Normal form games

Prisoner's dilemma
(normal form)

u	c	d
c	(-1,-1)	(-6,0)
d	(0,-6)	(-3,-3)

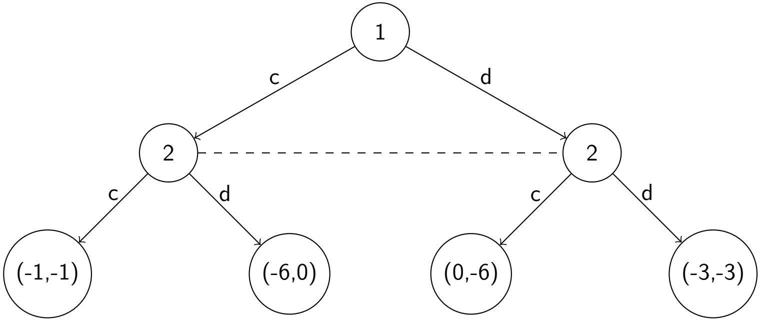
Best response

Action a is the best response to action b for payoff function v at world w if

$$w \models \forall \alpha . \exists x, y . \left(\begin{array}{c} (\langle a \rangle \top \wedge \langle \alpha \rangle \top) * (\langle b \rangle \top) \\ \to \\ v(\alpha b) \leq v(ab). \end{array} \right)$$

Extensive form games

Example (Prisoner's dilemma, extensive form)

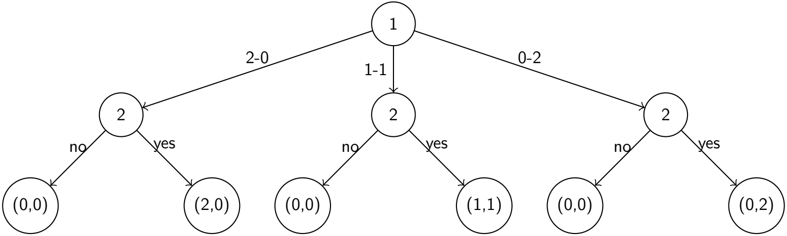


Extensive form games

- History-based semantics: worlds are sequences.
- Here, the histories are $c; c$, $c; d$, $d; c$, $d; d$
- Contrast to strategies in game theory:
  - Strategies specify the choice at *every* (distinguishable) decision point in the tree.

Extensive form games

Example (Sharing game, extensive form)



- Histories here are, for example, (2-0; no), and (1-1; yes).
- Strategies here are, for example (1-1, no, yes, no).

Extensive form games

Definition (Sub-game perfect equilibrium)

A strategy is a sub-game perfect equilibrium if it is the best response for all players at all sub-games.

- So (1-1, no, yes, no) is a sub-game perfect equilibrium, but (1-1, no, no, no) is not.

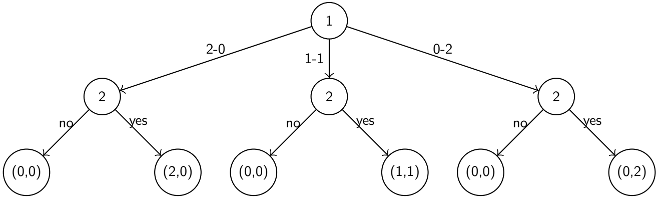
Extensive form games

Definition (Sub-game optimal history (proposed))

A history is sub-game optimal if it is empty, or, if both the following hold

1. The sub-game optimal property holds at the next stage of the history, and,
2. There exists no (distinguishable) alternative history that the (current) decision maker (weakly) prefers, where the sub-game-optimal property holds.

Extensive form games



- Consider the histories (2-0; yes), and (1-1; yes).
  - The first agent prefers the history (2-0; yes) to (1-1; yes).
  - However, at the second decision point, the history (no) is weakly preferred to the history (yes).
  - Hence (yes), at the second decision point, is not a sub-game optimal history.
- The history (2-0, yes) *is not* a sub-game optimal history.
- The history (1-1, yes) *is* a sub-game optimal history.

Extensive form games

Proposed logical components to express sub-game optimality of a history:

1. Non-commutative substructural connectives.
   - Conjunction, $\phi \blacktriangleright \psi$, to access *"the next stage of the history"*).
   - Unit, J, to represent *"empty"* histories.
2. Least fixed points, $\mu X.\phi$, to evaluate the optimality property at *"the next stage of the history"*).
3. A modality denoting the existence of distinguishable preference, for an agent i, $\triangle_i \phi$.
4. Propositions to denote which agent is the *"(current) decision maker"*.

Extensive form games

A history w is sub-game optimal, for a set of agents I, if

$$w \models \mu X. \left(J \vee \bigwedge_{I} \left(\begin{array}{c} \mathrm{owns}_i \\ \rightarrow \\ \left((\bigcirc X) \wedge \neg(\triangle_i(\bigcirc X)) \right) \end{array} \right) \right)$$

holds, where $\bigcirc\phi$ denotes that ϕ holds at the tail of the history.

Conclusion

- Modal commutative substructural logic describes normal form games well.
- Fixed-point non-commutative substructural logic describes extensive form games well.
- A combined logic may be useful.