Positive modal separation logics

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- Strong completeness w.r.t. Kripke frames
- Problem: incompletness in the presence of axioms, e.g. add $\Diamond\Diamond a \vdash \Diamond a$ to the logic and $\Box a \vdash \Box\Box a$ is valid but not derivable.

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- New semantics in terms of ordered Kripke frames with compatibility requirements between R and $\leqslant$. Valuations in *upsets*.
- Strong completeness preserved.
- This solves the problem:

$$\Diamond\Diamond p \models \Diamond p \text{ iff } (R;\geqslant) \text{ is transitive}$$

$$\Box p \models \Box\Box p \text{ iff } (R;\leqslant) \text{ is transitive}$$

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- All these semantics are related. Coalgebraic semantics: start with $R_\Diamond$, $R_\Box$ and use Interaction axioms to prove one R is enough.

Positive modal logic

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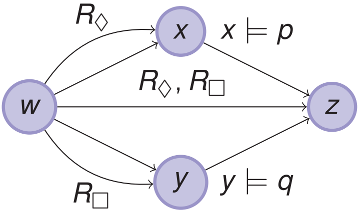

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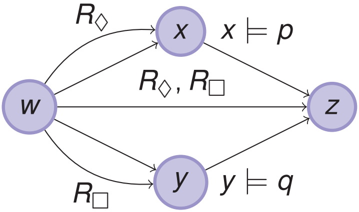

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Strong completeness.

Positive ML is strongly complete w.r.t. to Kripke frames with two convex relations $R_\Diamond, R_\Box$ and upset valuation validating Interaction axioms. Moreover:

$$w \models_{R_\Diamond \times R_\Box} a \qquad \text{iff} \qquad w \models_{(R_\Diamond \cap R_\Box) \times (R_\Diamond \cap R_\Box)} a$$

‘Separation logic’ as positive ML

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- Models: posets with convex binary relations and downset of ‘special points’:

$$w \models I \text{ if } w \in \mathcal{I}$$
$$w \models p * q \text{ if } \exists w R_*(x,y), x \models p \text{ and } y \models q$$
$$w \models p \mathrel{-\!\!*} q \text{ if } \forall w R_{-\!*}(x,y), x \models p \text{ implies } y \models q$$
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- Axioms: distribution laws of $*$, $\mathrel{-\!\!*}$, $\mathrel{*\!\!-}$ (think **K**) plus

1. $a * I \dashv\vdash a, I * a \dashv\vdash a$
2. $I \vdash a \mathrel{-\!\!*} a, I \vdash a \mathrel{*\!\!-} a$
3. $a * (b \mathrel{-\!\!*} c) \vdash (a * b \mathrel{-\!\!*}) c$
4. $(c \mathrel{*\!\!-} b) * a \vdash c \mathrel{*\!\!-} (a * b)$
5. $(a \mathrel{*\!\!-} b) * b \vdash a$
6. $b * (b \mathrel{-\!\!*} a) \vdash a$

‘Separation logic’ as positive ML

Strong completeness of ‘separation logic’

Positive ‘separation logic’ is strongly complete w.r.t. Kripke frames with convex ternary relations R_*, R_{-*}, R_{*-} validating its axioms. This means that it is complete w.r.t. to Kripke frames with a single convex ternary relation R

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Much more general result: residuation is preserved under canonical extension on boolean algebras, distributive lattices, semi-lattices and even posets!

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- Take $W = \{f : \mathbb{N}_+ \rightharpoonup_f \mathbb{N}\}$ with $f \leqslant g$ whenever $f = g \restriction \mathrm{dom} f$, $\mathcal{I} = \{\mathrm{Id}_U \mid U \in \mathcal{P}_f(\mathbb{N}_+)\}$ and $f\,R\,(g, h)$ iff $\mathrm{dom} g \cap \mathrm{dom} h = \emptyset$, $g \leqslant f$, $h \leqslant f$

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- For $(P, \circ, I)$ a partial monoid, take $W = P$ with $a \leqslant b$ if $\exists c, a \circ c = b$, $\mathcal{I} = I$ and $a\,R\,(b, c)$ iff $b * c \leqslant a$

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- For $(P, \leqslant, \circ, I)$ an ordered partial monoid: same as above with native order.
- For any set X, take $W = \{S \subseteq X \times X\}$ with $\leqslant$ given by $\subseteq$, $\mathcal{I} = \{\mathrm{Id}_U \mid U \subseteq X\}$ and $S\,R\,(T_1, T_2)$ whenever $T_1; T_2 \subseteq S$.

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Strong completeness is modular

Let Σ_1, Σ_2 be two signatures and $\mathsf{Ax}_1, \mathsf{Ax}_2$ be sets of *canonical* axioms in $\mathcal{L}_{\Sigma_1}$ and $\mathcal{L}_{\Sigma_2}$ which include distribution laws, then $\mathcal{L}_{\Sigma_1} \oplus \mathcal{L}_{\Sigma_2}/\{\mathsf{Ax}_1 \cup \mathsf{Ax}_2\}$ is strongly complete w.r.t. to Kripke frames with convex *n*-ary relations R_σ, $\sigma \in \Sigma_1 \cup \Sigma_1$ validating the axioms in $\mathsf{Ax}_1 \cup \mathsf{Ax}_2$.

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  - Encode models of time: e.g. the smallest temporal logic $K_{+}^{P,F}4_P4_F C_P C_F\oplus$SPL where

$$C_P = \{a \vdash [P]\langle F\rangle a,\ \ \langle p\rangle[F]a \vdash a\}, C_F = \{a \vdash [F]\langle P\rangle a,\ \ \langle F\rangle[P]a \vdash a\}$$

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  - Combine labels with a grammar and encode the grammar as axioms. For example $l ::= \pi \mid l; l$ and

    $$\langle l_1 \rangle \langle l_2 \rangle p \dashv\vdash \langle l_1 ; l_2 \rangle p, \qquad [l_1][l_2]p \dashv\vdash [l_1 ; l_2]p$$

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  - Beyond fusions: introducing modal-separation interaction e.g. $l ::= \pi \mid l; l \mid l \parallel l$

$$\langle l_1 \parallel l_2 \rangle p \dashv\vdash \langle l_1 \rangle p * \langle l_2 \rangle p, \qquad [l_1 \parallel l_2]p \dashv\vdash [l_1]p * [l_2]p$$

    Modularity does not provide strong completeness anymore, but canonicity of all the axioms does.

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- Using canonicity we can build strongly complete positive modal separation logics where labels interact with $*$

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- ... it can all be done with $\bot$ and $\top$ too ... and negations as well ...

Thank you.